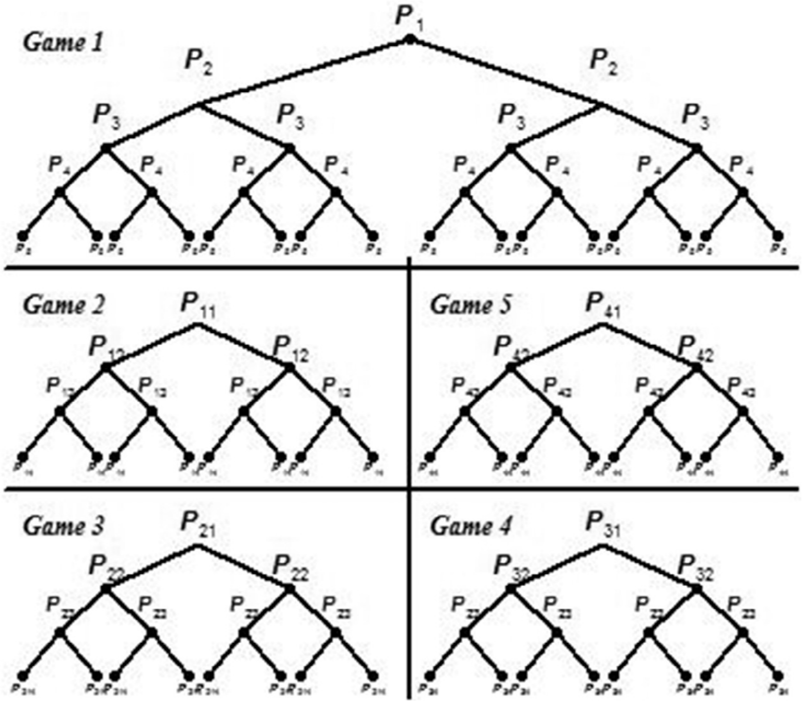

The Beautiful Mind—John Forbes Nash, Jr.—His works in Game Theory, geometry, etc. have provided insight into the factors that govern chance and events in our daily lives.

Chapter 12

Game Theory

The Joy of Finite Mathematics

JOHN VON NEUMANN (1903-1957)

John von Neumann, one of the foremost twentieth-century mathematicians, was born in Budapest, Hungary on December 28, 1903. He was a general scientific prodigy, in the mode of some of his great predecessors. He received his formal education at the University of Berlin, the Technical Institute in Zurich, and the University of Budapest, where he earned his Ph.D. in mathematics in 1926 at the age of twenty.

Von Neumann taught at the University of Hamburg from 1926 until 1929. He left Germany in 1930, just before the Second World War, taking refuge in this country, where he accepted a professorship in mathematical physics at Princeton University. He became professor of mathematics at the Institute for Advanced Study at Princeton, New Jersey, when it was founded in 1933, retaining this position until his death in 1957.

Von Neumann was one of the founders of game theory. In 1944 he collaborated with Oskar Morgenstern on a book, Theory of Games and Economic Behavior. This important and initial comprehensive treatment of game theory offered a new approach to the study of economic behavior through the use of game-theoretic methods.

The development of the first electronic computers at the institute was directed by von Neumann. He initiated the concept of stored programs in the computer—that programs and data should be treated similarly. This was one of the major breakthroughs in computer development. Among the computers for which von Neumann is credited are the MANIAC, NORC, and ORDVAC.

Von Neumann served on the U.S. Atomic Energy Commission and consulted on the Atomic Bomb Project at Los Alamos. In 1956, he received the $50,000 Enrico Fermi award for his outstanding work on computer theory, design, and construction.

12.1 INTRODUCTION TO GAME THEORY

Game theory is a relatively new branch of mathematics; it is the study of the rational behavior of people in conflict situations. The conflict may be between individuals involved in a game of chance, between teams engaged in an athletic contest, between nations engaged in war, or between firms engaged in competition for a share of the market for a certain product. In our study of game theory, we shall refer to each contestant as a player and restrict our investigations to games employing two players. In general, however, we need not restrict ourselves to conflicts generated by two players only; such games can involve any number of players.

The two players involved in a game have precisely opposite interests. Each player has a specified number of actions from which to choose. However, the action taken by one of the players at a particular stage of the game is not known to the other player. Each of the players has a certain objective in mind, and he attempts to choose his actions so as to achieve this aim.

Reality is broken: why games make us better and how this changes the world.

Jane McGonigal

Definition 12.1.1 Game Theory

Game theory is concerned with the analysis of human behavior in conflict situations. In our brief introduction to the subject, we have considered only games between two persons (players); however, they may involve any number of players, teams, companies and so forth.

The initial study of game theory was conducted independently by John von Neumann and Emil Borel during the 1920s. After World War II, von Neumann and Oskar Morgenstern formulated the subject area as an independent branch of mathematics. The close relationship between systems of equations and game theory will become apparent as we proceed with our study.

12.2 THE MATRIX GAME

All reality is a game.

I. Banks

Consider a modified version of the game "matching pennies." Player P_1 puts a coin either heads up, H, or tails up, T. A second Player P_2 without knowing P_1's choice calls either "heads" or "tails." Player P_1 will pay Player P_2 \$5 if P_1 shows H and P_2 chooses H; Player P_1 will pay Player P_2 \$3 if both have chosen T. However, if Player P_2 guesses incorrectly, he must pay P_1 \$4 (that is, if P_1 shows H and P_2 guesses T, or if P_1 shows T and P_2 guesses H Player P_2 must pay P_1 \$4).

This game is a two-person game since there are two players involved. It can be displayed in the form of a matrix as follows:

$$\begin{array}{ccc} & & P_2 \\ & & \begin{array}{cc} H & T \end{array} \\ P_1 & \begin{array}{c} H \\ T \end{array} & \begin{bmatrix} -5 & 4 \\ 4 & -3 \end{bmatrix} \end{array}$$

The positive entries of the matrix denote the gains of Player P_1, and his losses to Player P_2 are the negative entries of the array. We designate P_1, the role player and P_2 the column player. Player P_1's two choices are each associated with a row of the matrix. Player P_2's choices are each associated with a column of the matrix. Thus, if P_1 chooses row one and P_2 chooses column one, then P_1 pays P_2 \$5; instead, if P_2 selects column two, then P_2 pays P_1 \$4.

Definition 12.2.1 Game Theory/Payoff Matrix

The states of the game are given in the form of a matrix called the **game matrix** or **payoff matrix**.

This game is a *zero-sum game* because whatever is lost or gained by P_1 is gained or lost by P_2. The matrix representation of the game is called the *matrix game*. Each entry of the matrix game represents the payoff to either player P_1 or P_2. Thus a matrix game is also referred to as the payoff matrix.

Definition 12.2.2 Two-person Zero Game

A **two-person zero-sum game** is a game played by two opponents with opposing interest and such that the payoff to one player is equal to the loss of the other.

The problem facing each player is what choice to make so that it will be in his best interest. That is, should P_1 select row one or row two? Should P_2, not knowing P_1's choice, select column one or column two? Before we proceed to discuss the method for choosing optimally, we shall summarize the basic concept and terminology constituting a two-person zero-sum game:

(a) Two players, P_1 and P_2, are engaged in a conflict of interest. For example, in the game of "matching pennies," P_1 wants to maximize his winnings while P_2 wants to minimize his losses.

(b) Each player has at his disposal a set of instructions regarding the action to take in each conceivable position of the game. We shall refer to these instructions as *strategies*. (In the game given above each player has two strategies H and T.)

(c) Associated with each strategy employed by P_1, there is a certain payoff. The set of all strategies will result in an array of payoffs known as the matrix game or payoff matrix.

For instance, in "matching pennies," the payoff matrix was

$$\begin{array}{ccc} & & \text{Strategies of } P_2 \\ & & \overbrace{\;H \quad T\;} \\ \text{Strategies of } \{P_1 & \begin{array}{c} H \\ T \end{array} & \begin{bmatrix} -5 & 4 \\ 4 & -3 \end{bmatrix} \end{array}$$

(d) The objective of P_1 is to utilize his strategies so as to maximize his winnings; P_2's objective is to select his strategies so as to minimize his losses.

(e) Such games are called zero-sum games or strictly competitive games, since the sum of the amounts won by the two players is always zero.

Note: The winnings of one player are equal to the losses of the other.

> *Like all mathematics, game theory is a tautology whose conclusions are true because they are contained in the premises.*
>
> ***R.A. Epstein***

Example 12.2.1 Payoff Matrix

Consider a two-person zero-sum game, the payoff matrix of which is given by

$$\begin{array}{ccc} & & P_2 \\ & & \begin{array}{ccc} \gamma_1 & \gamma_2 & \gamma_3 \end{array} \\ P_1 & \begin{array}{c} \beta_1 \\ \beta_2 \end{array} & \begin{bmatrix} -2 & 1 & 3 \\ 6 & -4 & 5 \end{bmatrix} \end{array}$$

Here, there are two strategies available to Player P_1; namely β_1 and β_2, and three strategies, γ_1, γ_2, γ_3, available to Player P_2. The entries in the matrix denote winnings of Player P_1. Recall that negative entries mean that Player P_1 will pay P_2. In the game P_1 chooses one of his two strategies, β_1 or β_2, and simultaneously, Player P_2, without knowing P_1's choice selects γ_1, γ_2, γ_3. The intersection of the row corresponding to P_1's choice and the column corresponding to P_2's selection gives the payoff of this play in the game. For example, if P_1 chooses strategy β_2 and P_2 selects γ_3, then P_1 wins \$5. If P_1 chooses β_1 and P_2 selects γ_1, then P_1 receives \$-2; that is, he must pay Player P_2 \$2.

Here, we have the question of importance: What strategy should P_1 choose to maximize his winnings? At the same time, what should P_2's choice be to minimize his losses? The selection of strategies will be the focus of the remaining discussions in this chapter.

12.3 STRICTLY DETERMINED GAME: THE SADDLE POINT

Games lubricate the body and mind.

Benjamin Franklin

Let us consider a two-person game, the payoff matrix of which is given by

$$\begin{array}{cc} & P_2 \\ & \begin{array}{ccc} \gamma_1 & \gamma_2 & \gamma_3 \end{array} \\ P_1 \begin{array}{c} \beta_1 \\ \beta_2 \\ \beta_3 \end{array} & \begin{bmatrix} 3 & 7 & 2 \\ 4 & 5 & 8 \\ 2 & 3 & 4 \end{bmatrix} \end{array}$$

Here, P_1 wants to maximize his payoff, while P_2 wishes to minimize it. Before we proceed to determine the optimal strategy for the game, we shall employ the concept of row and column domination to simplify the game. That is, each of the elements of row 2 is greater than the corresponding element of row three ($4>2$, $5>3$, $8>4$). In view of P_1's objectives of maximizing his payoff, he can always do better by choosing β_2 instead of β_3. Thus, we can eliminate row 3 of the game matrix. P_2's aim is to minimize his payoff to P_1. Assuming that P_1 will be playing rationally, it is clear, looking at the three strategies available to P_2, that he will not choose strategy γ_2 (second column). This is because no matter what strategy P_1 uses, P_2 can do better than γ_2 by selecting γ_1, since each entry in the first column is smaller than the corresponding entry in the second column. Thus, the second column can be eliminated from the game matrix since it is dominated by column one. That is, if Player P_2 plays intelligently, he will never consider strategy γ_2.

At this point, the payoff matrix is reduced to

$$\begin{array}{cc} & P_2 \\ & \begin{array}{cc} \gamma_1 & \gamma_2 \end{array} \\ P_1 \begin{array}{c} \beta_1 \\ \beta_2 \end{array} & \begin{bmatrix} 3 & 2 \\ 4 & 8 \end{bmatrix} \end{array}$$

We shall not attempt to analyze the game. If P_1 uses strategy β_1, the worst that can happen is that P_2 will select strategy γ_3 and P_1 will only gain \$2. If P_1 uses strategy β_2, the worst that can happen is that P_2 will select strategy γ_1 and P_1 will only gain \$4. The best approach to the game by Player P_1 is to aim at the larger of these minimum payoffs. Thus, P_1 must choose strategy β_2, which will result in the *maximum* of the *minimum* amounts he stands to gain.

The method of selecting the best strategy for P_1 is called the *maximin decision*, which is simply the maximum of the row minimum.

Rule 12.3.1 Maximin/Minimax

The search for the saddle point of a matrix game, if one indeed exists, can be made in three steps:

Step 1 Write the minimum entry of each row at the end of that row and write the maximum entry or each column at the end of the column.

Step 2 Identify the maximin strategy for Player P_1 by circling the largest of the row minimum. Similarly, circle the smallest of the column maximum to identify the minimax strategy for Player P_2.

Step 3 The game has a saddle point if and only if the maximin and minimax are equal to the same number, and that number also appears at the intersection in the matrix of maximin (row) and minimax (column).

Now let us consider P_2. If he uses strategy γ_1, the worst that can happen is that he will lose \$4 if P_1 employs strategy β_2. If he selects strategy γ_3, the worst that can happen is that he will lose \$8. Thus, it is to his advantage to choose γ_1

in that this choice will minimize his maximum loss to $\boldsymbol{P_1}$. This method of selecting the strategy for Player $\boldsymbol{P_2}$ is called the minimax decision, which is simply the minimum of the column maximum.

The selection of the preceding strategies can be displayed in the form of a table as follows:

P_2

		γ_1	γ_3	***Row minimum***
P_1	β_1	3	2	2
	β_2	4	8	4
	Column maximum	4	8	

Thus, the maximum decision for $\boldsymbol{P_1}$ is the strategy resulting in the maximum of the row minimum, which is $\boldsymbol{\beta_2}$, and the minimax decision for $\boldsymbol{P_2}$ is the strategy resulting in the minimum of the column maximum, which is $\boldsymbol{\gamma_1}$. In the present example, $\boldsymbol{P_1}$'s maximum decision results in a payoff of \$4, which coincides with $\boldsymbol{P_2}$'s minimax decision, also resulting in a payoff of \$4. When this fact occurs, we say that the matrix game possesses a *saddle point or point of equilibrium*. We call this entry in the game matrix the *value of the game*. A matrix game possessing a point of equilibrium is said to be *strictly determined*. We shall summarize these important terms in the following definition:

Game Theory is the Art and Science of Negotiation Prologue, …

H. Raiffa

Definition 12.3.1 Strictly Determined

A matrix game is said to be **strictly determined** if and only if there is an entry in the payoff matrix that is the smallest entry in its row and also the largest entry in its column. This entry is called the **saddle point** or **equilibrium point** and is the value of the game.

The name *saddle point* is derived from the property of the point as the minimum in its row and the maximum in its column, as visualized by the point (β_i, γ_j) in Figure 12.1.

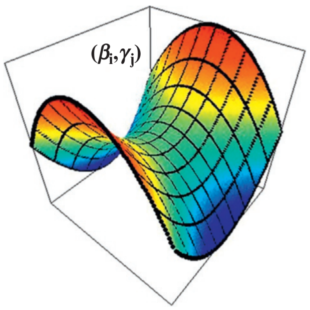

FIGURE 12.1 The saddle point.

We shall illustrate our discussion by considering some examples.

Example 12.3.1 Saddle Point

Obtain the value of the following matrix game:

$$\begin{array}{cc|ccc} & & & P_2 & \\ & & \gamma_1 & \gamma_2 & \gamma_3 \\ & \beta_1 & \$-3 & \$1 & \$4 \\ P_1 & \beta_2 & \$4 & \$-3 & \$5 \\ & \beta_3 & \$6 & \$2 & \$7 \end{array}$$

Solution

Here again, the aim of P_1 is to maximize his winnings and that of P_2, to minimize his losses. Inspecting the payoff matrix, we see that the second row is dominated by the third row. This is, each entry in the third row is larger than the corresponding entry in the second row. Thus, if P_1 plays intelligently, he will never employ strategy β_2, so we can eliminate row 2. Similarly, column 3 is dominated by column 2. Thus, if P_2 plays intelligently, he will not use strategy γ_3. Hence, the payoff matrix reduces to

$$\begin{array}{ccc} & & P_2 \\ & & \begin{array}{cc}\gamma_1 & \gamma_2\end{array} \\ P_1 & \begin{array}{c}\beta_1 \\ \beta_2\end{array} & \begin{bmatrix}\$-3 & \$1 \\ \$6 & \$2\end{bmatrix}\end{array}$$

Applying the *maximin* and *minimax* method we have

P_2

		γ_1	γ_2	*Row minimum*
P_1	β_1	\$–3	\$–1	\$–3
	β_3	\$6	\$2	\$2
	Column maximum	\$6	\$2	

Thus, the best strategy for P_1 to employ is β_3. This will assure him a gain of \$2, provided player P_2 plays intelligently and uses strategy γ_2 which will cost him only \$2. Since the *maximin and minimax* decisions result in the same amount of payoff; namely, \$2, this is a *saddle point*. The game is *strictly determined* since it possesses a *point of equilibrium*. The *value of the game* is \$2.

In attempting to find the best strategies of the players, we need not employ the row and column dominance technique to reduce the payoff matrix. We can directly employ the maximin-minimax procedure.

Game theory is a study of strategic decision making.

Example 12.3.2 Best Strategies

Find the best strategies for Players P_1 and P_2 which will lead to the value of the game if the payoff matrix is given by

$$\begin{array}{ccc} & & P_2 \\ & & \begin{array}{cccc}\gamma_1 & \gamma_2 & \gamma_3 & \gamma_4\end{array} \\ P_1 & \begin{array}{c}\beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4\end{array} & \begin{bmatrix}\$10 & \$0 & \$24 & \$-14 \\ \$12 & \$5 & \$6 & \$18 \\ \$-4 & \$-3 & \$ & \$-2 \\ \$12 & \$-7 & \$6 & \$5\end{bmatrix}\end{array}$$

Solution

Again, the aim of P_1 is to find the best strategy to maximize his winnings. P_2 wishes to employ the strategy to minimize his losses. Utilizing the maximin-mimimax procedure, we construct the table below. Thus, Player P_1 should employ strategy β_2 to maximize the minimum amount he stands to win. Player P_2 should employ strategy γ_2 to minimize the maximum amount he stands to lose. It is clear that the value of the game is \$5.

Continued

Solution—cont'd

P_2

P_1		γ_1	γ_2	γ_3	γ_4	*Row minimum*
	β_1	**\$10**	**\$0**	**\$24**	**\$–14**	**\$–14**
	β_2	**\$12**	**\$5**	**\$6**	**\$18**	**\$5**
	β_3	**\$–4**	**\$–3**	**\$16**	**\$–2**	**\$–4**
	β_4	**\$12**	**\$–7**	**\$6**	**\$5**	**\$–7**
	Column maximum	**\$12**	**\$5**	**\$24**	**\$18**	

The solutions of strictly determined games possess certain properties that are optimal to both players provided that they play intelligently. If one of the players deviates from the procedure leading to the saddle point, the other player may increase his profit. For instance, suppose that in Example 12.3.2, Player P_1 uses his best strategy, β_2, but that P_2 plays irrationally by using strategy γ_4. This will result in P_1 increasing his payoff from \$5 to \$18. If both players deviate from their optimal strategies, then, needless to say, one of them will suffer.

In this section, we have discussed two-person games that are strictly determined; however, there are games that do not possess a point of equilibrium. Such games will be studied in Section 12.4.

PROBLEMS

12.3.1 Find the best strategies for players P_1 and P_2 in the matrix games given by

(a)
$$\begin{array}{cc} & P_2 \\ & \begin{array}{cc}\gamma_1 & \gamma_2\end{array} \\ P_1 \begin{array}{c}\beta_1\\ \beta_2\end{array} & \begin{bmatrix} \$4 & \$6 \\ \$5 & \$8 \end{bmatrix}\end{array}$$

(b)
$$\begin{array}{cc} & P_2 \\ & \begin{array}{ccc}\gamma_1 & \gamma_2 & \gamma_3\end{array} \\ P_1 \begin{array}{c}\beta_1\\ \beta_2\end{array} & \begin{bmatrix} \$4 & \$10 & \$6 \\ \$-3 & \$5 & \$-2 \end{bmatrix}\end{array}$$

The aim of P_1 is to maximize his payoff, P_2's objective is to minimize his losses.

12.3.2 Find the saddle point of the following matrix games and interpret their meanings:

$$\text{(a)} \begin{bmatrix} \$6 & \$0 & \$-5 \\ \$3 & \$2 & \$6 \\ \$-7 & \$1 & \$-11 \end{bmatrix} \quad \text{(b)} \begin{bmatrix} \$-3 & \$4 & \$-2 \\ \$0 & \$1 & \$5 \\ \$-2 & \$16 & \$-13 \end{bmatrix}$$

12.3.3 Determine whether or not the following matrix games are strictly determined:

$$\text{(a)} \begin{bmatrix} \$-2 & \$3 & \$-1 \\ \$-5 & \$1 & \$-16 \end{bmatrix}, \quad \text{(b)} \begin{bmatrix} \$9 & \$2 & \$4 \\ \$7 & \$1 & \$7 \\ \$-1 & \$3 & \$14 \end{bmatrix}$$

12.3.4 Construct a 3×3 matrix game, the value of which is \$10.

12.3.5 Construct a two-person zero-sum matrix game in which the payoff to the maximizing player is negative.

12.4 GAMES WITH MIXED STRATEGIES

Evolution and the Theory of Games
J. Maynard Smith

As we mentioned in the previous section, not all two-person zero-sum games possess a saddle point. We shall now turn our attention to those games that are not strictly determined. Consider the following two-player game:

$$\begin{array}{ccc} & & P_2 \\ & & \begin{array}{cc}\gamma_1 & \gamma_2\end{array} \\ P_1 & \begin{array}{c}\beta_1 \\ \beta_2\end{array} & \begin{bmatrix} \$8 & \$3 \\ \$4 & \$16 \end{bmatrix} \end{array}$$

Applying the procedure we discussed in our search for a saddle point, we can write the game as

P_2

		γ_1	γ_2	*Row minimum*
P_1	β_1	**\$8**	**\$3**	**\$3**
	β_2	**\$4**	**\$16**	**\$4**
	Column maximum	**\$8**	**\$16**	

If Player P_1 chooses his maximin strategy, then his maximin payoff will be \$4. If Player P_2 employs his minimax strategy, then his minimax payoff will be \$8. Thus, the maximin payoff of P_1 and the minimax payoff of P_2 are not equal—the matrix game does not possess a point of equilibrium. For such a game, it is difficult to see what either player should do.

Let us consider the case in which Player P_2 employs his minimax strategy, γ_1. If Player P_1 suspects P_2‘s choice, he can then select strategy β_1 instead of his maximin strategy, β_2, to increase his payoff from \$4 to \$8. However, if Player P_1 continues to employ strategy β_1, P_2 will detect this and can shift from his minimax strategy, γ_1, to strategy γ_2 decreasing his losses from \$8 to \$3. After a while, Player P_1 will likely become aware of P_2's strategy and can shift his choice from β_1 to β_2, thus increasing his payoff from \$3 to \$16. It is clear, then, in games without a saddle point, that a player should not stick with a particular strategy, but rather mix his strategies in such a way that the opposite player will not easily be able to predict his choice.

Mixing strategies can be done by some random mechanism to assure that the opposing player in the game will not discover the pattern of moves. For example, the random mechanism can be a fair coin. Player P_1 flips a coin; if a head appears, he chooses strategy β_1 with probability p; if a tail occurs, he chooses strategy β_2 with probability $1-p$. Note that $p+(1-p)=1$; that is, we are certain that at each move one of the two strategies will be chosen. Thus, the random mechanism mixes the strategies of the players. Games in which each player's strategies are mixed for lack of a saddle point are called *mixed-strategy games*.

Now, suppose that in mixed-strategy games, we assign to each player a certain probability for choosing each of his strategies. What we want to know here is how we can characterize the payoff of a game? In strictly determined games the payoff of the game was defined by the maximin and minimax strategies. However, in the present case we do not know which strategies are being used by the players and we cannot define the payoff if the game is played only once. However, if the game is played a number of times, by utilizing the frequency with which each strategy is used, we can obtain the expected payoff or central tendency of the matrix game. For example, suppose that in the matrix game on page 504 Player P_1 chooses his strategies β_1 and β_2 with probabilities 0.6 and 0.4, respectively. Similarly, player P_2 selects his strategies γ_1 and γ_2 with probabilities 0.7 and 0.3, respectively. (Shortly we shall be discussing the best way of arriving at these probabilities.) The expected payoff of the matrix game is given by evaluating the following expression:

$$[0.6 \;\; 0.4] \cdot \begin{bmatrix} 8 & 3 \\ 4 & 10 \end{bmatrix} \begin{bmatrix} 0.7 \\ 0.3 \end{bmatrix}$$

First, we multiply the vector [**0.6 0.4**] and the 2×2 matrix game $\begin{bmatrix} 8 & 3 \\ 4 & 16 \end{bmatrix}$ that is,

$$[0.6 \;\; 0.4] \cdot \begin{bmatrix} 8 & 3 \\ 4 & 16 \end{bmatrix} = [4.8+1.6 \;\; 1.8+6.4] = [6.4 \;\; 8.2].$$

Next, we multiply the row vector [**6.4 8.2**] by the column vector $\begin{bmatrix} 0.7 \\ 0.3 \end{bmatrix}$ to obtain the expected payoff,

$$\begin{aligned}[0.6 \;\; 0.4 \;\; 8.2] \cdot \begin{bmatrix} 0.7 \\ 0.3 \end{bmatrix} &= (6.4)(0.7) + (8.2)(0.3) \\ &= 4.48 + 2.46 \\ &= \$6.94\end{aligned}$$

Thus, the expected payoff which favors Player P_1, is \$6.94. That is, in the long run and under the specified probabilities of selecting the strategies, P_1's expected gain is \$6.94.

In general, the expected payoff of a 2×2 matrix game is obtained as follows: Let P_1 and P_2 be the probabilities that Player P_1 selects his strategies, β_1 and β_2 respectively. We shall refer to these probabilities as *strategy probabilities* of Player P_1 and write them as a row vector $[p_1 p_2]$. Similarly, we shall denote the strategy probabilities for Player P_2 by the column vector $\begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$. Note that $p_1 + p_2 = 1$ and $q_1 + q_2 = 1$. The expected payoff of the matrix game $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ denoted by E is defined by

$$E = [p_1 \;\; p_2] \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$$

Example 12.4.1 Expected Payoff

Obtain the expected payoff of the matrix game,

$$\begin{array}{ccc} & & P_2 \\ & & \begin{array}{cc} \gamma_1 & \gamma_2 \end{array} \\ P_1 & \begin{array}{c} \beta_1 \\ \beta_2 \end{array} & \begin{bmatrix} 4 & -3 \\ -5 & 7 \end{bmatrix} \end{array}$$

If Player P_1 and P_2 decide on selecting their strategies with probabilities [1/2 ½] and [1/3 2/3], respectively.

Solution

The expected payoff of this game is

$$E=\begin{bmatrix}\frac{1}{2} & \frac{1}{2}\end{bmatrix}\begin{bmatrix}4 & -3\\ -5 & 7\end{bmatrix}\begin{bmatrix}\frac{1}{3}\\ \frac{2}{3}\end{bmatrix}$$

$$=\begin{bmatrix}-\frac{1}{2} & \frac{4}{2}\end{bmatrix}\begin{bmatrix}\frac{1}{3}\\ \frac{2}{3}\end{bmatrix}$$

$$=\frac{7}{6}$$

Thus, the expected payoff is in favor of P_1. Note that if the actual value of E had been negative, then the game would have been in favor of Player P_2.

For higher order matrix games, say $k\times k$, given by

$$\begin{array}{c} P_2 \\ \begin{array}{c|cccc|} & \gamma_1 & \gamma_2 & \cdots & \gamma_k \\ \beta_1 & a_{11} & a_{12} & \cdots & a_{1k} \\ \beta_2 & a_{21} & a_{22} & \ldots & a_{2k} \\ \vdots & \vdots & \vdots & & \vdots \\ \beta_k & a_{k1} & a_{k2} & \cdots & a_{kk} \end{array} \end{array}$$

with strategy probabilities for P_1 and P_2 given by

$$[p_1 \ \ p_2 \ \ \cdots \ \ p_k] \quad \text{and} \quad \begin{bmatrix}q_1\\ q_2\\ \vdots\\ q_k\end{bmatrix}$$

respectively, the expected payoff of the game is

$$E=[p_1 \ \ p_2 \ \ \cdots \ \ p_k]\begin{bmatrix}a_{11} & a_{12} & \ldots & a_{1k}\\ a_{21} & a_{22} & \ldots & a_{2k}\\ . & . & & .\\ a_{21} & a_{22} & \ldots & a_{2k}\end{bmatrix}\begin{bmatrix}q_1\\ q_2\\ \vdots\\ q_k\end{bmatrix}$$

with $p_1+p_2+\cdots+p_k=1$ and $q_1+q_2+q_k=1$.

Example 12.4.2 Expected Payoff

Obtain the expected payoff of the matrix game,

$$\begin{array}{c} P_2 \\ \begin{array}{cc|rrr|} & \beta_1 & \gamma_1 & \gamma_2 & \gamma_3 \\ P_1 & \beta_2 & 4 & -3 & 0 \\ & \beta_3 & 2 & 1 & -4 \\ & & -10 & -3 & 1 \end{array} \end{array}$$

if the strategy probabilities of P_1 and P_2 are given by

$$\begin{bmatrix}\frac{1}{3} & \frac{1}{3} & \frac{1}{3}\end{bmatrix} \quad \text{and} \quad \begin{bmatrix}\frac{1}{2}\\ \frac{1}{4}\\ \frac{1}{4}\end{bmatrix}$$

respectively.

Solution

The expected payoff of the 3×3 matrix game is

$$E=\begin{bmatrix}\frac{1}{3} & \frac{1}{3} & \frac{1}{3}\end{bmatrix}\cdot\begin{bmatrix}4 & -3 & 0\\ 2 & 1 & -4\\ -10 & -3 & 1\end{bmatrix}\begin{bmatrix}\frac{1}{2}\\ \frac{1}{4}\\ \frac{1}{4}\end{bmatrix}$$

$$=\begin{bmatrix}-\frac{4}{3} & -\frac{5}{3} & -1\end{bmatrix}\begin{bmatrix}\frac{1}{2}\\ \frac{1}{4}\\ \frac{1}{4}\end{bmatrix}$$

$$=-\frac{4}{3}$$

Thus, the expected payoff of the game is –4/3 and it favors Player P_2.

There is still a basic question remaining: What is the best way for Player P_1 to choose his strategy probabilities so that his expected payoff will be maximum? To answer this question we proceed as follows. Consider a two-person zero-sum game given by

$$\begin{array}{ccc} & & P_2 \\ & & \begin{array}{cc}\gamma_1 & \gamma_2\end{array} \\ P_1 & \begin{array}{c}\beta_1\\ \beta_2\end{array} & \begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}\end{array}$$

and P_1's strategy probabilities by $[p_1\, p_2]$. Player P_1 wants to choose the probabilities p_1 and p_2 to select his strategies so as to maximize the expected payoff E.

That is,

$$E\le[p_1 \quad p_2]\begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}$$

Which can be written as

$$a_{11}p_1+a_{21}p_2\ge E$$
$$a_{12}p_1+a_{22}p_2\ge E,$$

such that

$$p_1+p_2=1.$$

Treating these inequalities as equalities and solving the three equations we obtain

$$p_1=\frac{a_{22}-a_{21}}{a_{11}-a_{12}-a_{21}+a_{22}}$$

and

$$p_2=\frac{a_{11}-a_{12}}{a_{11}-a_{12}-a_{21}+a_{22}}$$

Similarly, the aim of P_2 is to select his strategy probabilities, q_1 *and* q_2 in such a manner as to minimize the expected payoff. That is,

$$E=\begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}\begin{bmatrix}q_1\\ q_2\end{bmatrix}$$

which can be stated as

$$a_{11}q_1+a_{12}q_2\ge E$$
$$a_{21}q_1+a_{22}q_2\le E,$$

such that

$$q_1+q=1.$$

Solution—cont'd

Solving the inequalities as if they were equalities, we have

$$q_1 = \frac{a_{22} - a_{12}}{a_{11} - a_{12} - a_{21} + a_{22}}$$

$$q_2 = \frac{a_{11} - a_{21}}{a_{11} - a_{12} - a_{21} + a_{22}}$$

and

$$E = [p_1 \ \ p_2] \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} [q_1 \ \ q_2]$$

$$= \frac{a_{11} - a_{21}}{a_{11} - a_{12} - a_{21} + a_{22}}$$

> *Baseball, it is said, is only a game. True, … and the Grand Canyon is only a hole in Arizona. Not all holes, or games are created equal.*
>
> ***George Will***

Definition 12.4.1 Optimal Strategies

The strategy probabilities $\{p_1, p_2\}$ and $\{q_1, q_2\}$, as defined above are called the **optimal strategies** for Players P_1 and P_2, respectively. The expected payoff E is called the value of the game.

Example 12.4.3 Expected Payoff

Consider "matching pennies" under the following rules. If Player P_1 has turned up H and P_2 calls H, then P_1 must pay P_2 \$6. However, if P_2 calls T, then P_2 must pay P_1 \$4. If P_1 has turned up T and P_2 calls H, then P_2 must pay P_1 \$8, and if P_2 calls T, then P_1 pays P_2 \$2. Find the optimal strategies for P_1 and P_2 and the value of the game.

Solution

The game matrix is given by

$$\begin{array}{cc} & P_2 \\ P_1 & \begin{bmatrix} \$-6 & \$4 \\ \$8 & \$-2 \end{bmatrix} \end{array}$$

It can be seen that the game does not have a saddle point. Thus, we must employ the mixed-strategies procedure. The optimal strategy for P_1 is

$$p_1 = \frac{a_{22} - a_{21}}{a_{11} - a_{12} - a_{21} + a_{22}} = \frac{-2-8}{-6-4-8-2} = \frac{1}{2}$$

and

$$p_2 = \frac{a_{11} - a_{12}}{a_{11} - a_{12} - a_{21} + a_{22}} = \frac{-6-4}{-6-4-8-2} = \frac{1}{2}$$

The optimal strategy for P_2 is

$$q_1 = \frac{a_{22} - a_{12}}{a_{11} - a_{12} - a_{21} + a_{22}} = \frac{-2-4}{-6-4-8-2} = \frac{3}{10}$$

and

$$q_2 = \frac{a_{11} - a_{21}}{a_{11} - a_{12} - a_{21} + a_{22}} = \frac{-6-8}{-6-4-8-2} = \frac{7}{10}$$

The expected payoff of the game corresponding to these optimal strategies is

$$E = \frac{a_{11}a_{22} - a_{12}a_{21}}{a_{11} - a_{12} - a_{21} + a_{22}} = \frac{(-6)(-2) - (4)(8)}{-6-4-8-2}$$
$$= \$1.00.$$

Continued

Solution—cont'd

Thus, Player P_1's optimal strategy is to choose row one with probability 1/2 and row two with the same probability. Player P_2's optimal strategy is to select column one with probability 3/10 and column two with probability 7/10. The value of the game is $1.00. That is, since E is positive, the game is favorable to Player P_1 in the long run.

Example 12.4.4 Expected Payoff

Find the optimal strategies for Players P_1 and P_2 for the matrix game given in Example 12.3.2. That is,

$$\begin{array}{cc} & P_2 \\ P_1 & \begin{bmatrix} 4 & -3 \\ -5 & 7 \end{bmatrix} \end{array}$$

Also obtain the value of the game.

Solution

We should mention here that in Example 12.4.1, we stated the strategy probabilities of [1/2 1/2] and [1/3 2/3] for P_1 and P_2, respectively, for purpose of illustrating the concept of expected payoff. Here we shall obtain the optimal strategies for each of the players.

The optimal strategy for P_1 is

$$p_1 = \frac{a_{22} - a_{21}}{a_{11} - a_{12} - a_{21} + a_{22}} = \frac{7+5}{4+3+5+7} = \frac{12}{19}$$

and

$$p_2 = \frac{a_{11} - a_{12}}{a_{11} - a_{12} - a_{21} + a_{22}} = \frac{4+3}{4+3+5+7} = \frac{7}{19}$$

The optimal strategy for P_2 is

$$q_1 = \frac{a_{22} - a_{12}}{a_{11} - a_{12} - a_{21} + a_{22}} = \frac{7+3}{4+3+5+7} = \frac{10}{19}$$

and

$$q_2 = \frac{a_{11} - a_{21}}{a_{11} - a_{12} - a_{21} + a_{22}} = \frac{4+5}{4+3+5+7} = \frac{9}{19}$$

The expected payoff of the game corresponding to these optimal strategies is

$$E = \frac{a_{11}a_{22} - a_{12}a_{21}}{a_{11} - a_{12} - a_{21} + a_{22}} = \frac{(4)(7) - (-3)(-5)}{4+3-5+7} = \frac{13}{19}$$

Thus, the optimal strategies for P_1 and P_2 are [12/19 7/19] and [10/19 9/19], respectively. The value of the game is 13/19 and since it is positive it favors Player P_1. That is, in the long run P_1 is expected to gain 13/19.

We should mention here that we can sometimes employ the row and column dominance relation to reduce a higher order matrix game so that the preceding formulas would be applicable. For example, consider the 3 × 3 matrix game given by

$$\begin{array}{cc} & P_2 \\ P_1 & \begin{bmatrix} 2 & 1 & 3 \\ 12 & 3 & 11 \\ 5 & 6 & 10 \end{bmatrix} \end{array}$$

Here, row one of the matrix is dominated by the second row, since each element of row two is greater than the corresponding element of row one. Similarly, since each element of column two is smaller than the corresponding element of column three, we conclude that column three is dominated by column two. Thus, the matrix game reduces to

Solution—cont'd

$$\begin{array}{cc} & \boldsymbol{P_2} \\ \boldsymbol{P_1} & \begin{bmatrix} 12 & 3 \\ 5 & 6 \end{bmatrix} \end{array}$$

and we can apply the mixed-strategies procedure to obtain optimal strategies for $\boldsymbol{P_1}$ and $\boldsymbol{P_2}$.

The optimal strategy for $\boldsymbol{P_1}$ is

$$p_1 = \frac{a_{22} - a_{21}}{a_{11} - a_{12} - a_{21} + a_{22}} = \frac{6-5}{12-3-5+6} = \frac{1}{10}$$

and

$$p_2 = \frac{a_{11} - a_{12}}{a_{11} - a_{12} - a_{21} + a_{22}} = \frac{12-3}{12-3-5+6} = \frac{9}{10}$$

The optimal strategy for $\boldsymbol{P_2}$ is

$$q_1 = \frac{a_{22} - a_{12}}{a_{11} - a_{12} - a_{21} + a_{22}} = \frac{6-3}{12-3-5+6} = \frac{3}{10}$$

and

$$q_2 = \frac{a_{11} - a_{21}}{a_{11} - a_{12} - a_{21} + a_{22}} = \frac{12-5}{12-3-5+6} = \frac{7}{10}$$

The expected payoff of the game corresponding to these optimal strategies is

$$E = \frac{a_{11}a_{22} - a_{12}a_{21}}{a_{11} - a_{12} - a_{21} + a_{22}} = \frac{(12)(6)-(3)(5)}{12-3-5+6} = \frac{57}{10}$$

The value of the game is 57/10 and it favors Player $\boldsymbol{P_1}$ because it is positive. This means that in the long run Player $\boldsymbol{P_1}$ is expected to gain 57/10.

PROBLEMS

12.4.1 Find the expected payoff of the matrix game given by

$$\begin{array}{ccc} & & \boldsymbol{P_2} \\ & & \begin{array}{cc} \gamma_1 & \gamma_2 \end{array} \\ \boldsymbol{P_1} & \begin{array}{c} \beta_1 \\ \beta_2 \end{array} & \begin{bmatrix} -6 & 12 \\ 4 & -10 \end{bmatrix} \end{array}$$

If Player $\boldsymbol{P_1}$ chooses strategy $\boldsymbol{\beta_1}$ 60% of the time and Player $\boldsymbol{P_2}$ selects strategy $\boldsymbol{\gamma_2}$ 70% of the time. Which player does the game favor?

12.4.2 Determine the expected payoff for the game

$$\begin{array}{cc} & \boldsymbol{P_2} \\ \boldsymbol{P_1} & \begin{bmatrix} 10 & -6 & 2 \\ -12 & 8 & 4 \\ 16 & -14 & -8 \end{bmatrix} \end{array}$$

for the strategy probabilities [**1/3 1/3 1/3**] and $\begin{bmatrix} {}^1/_4 \\ {}^1/_4 \\ {}^1/_2 \end{bmatrix}$ for Players $\boldsymbol{P_1}$ and $\boldsymbol{P_2}$, respectively.

12.4.3 Find the optimal strategies for Players $\boldsymbol{P_1}$ and $\boldsymbol{P_2}$, and the value of the following games:

$$\begin{array}{cc} & \boldsymbol{P_2} \\ \boldsymbol{P_1} & \begin{bmatrix} -4 & 6 \\ 8 & -2 \end{bmatrix} \end{array} \quad \text{and} \quad \begin{array}{cc} & \boldsymbol{P_2} \\ \boldsymbol{P_1} & \begin{bmatrix} 10 & -12 \\ -16 & 14 \end{bmatrix} \end{array}$$

12.4.4 In the game of matching pennies, consider the following situation: If Player P_1 turns up T and P_2 calls T, P_1 pays P_2 \$5. However, if Player P_2 calls H, then P_2 must pay P_1 \$8. Now if P_1 turns up H and P_2 calls T, then P_2 must pay P_1 \$10, but if P_2 calls H, then P_1 must pay P_2 \$12.

(a) Obtain the matrix of the game.

(b) Determine the optimal strategies for Players P_1 and P_2.

(c) Find the value of the game.

(d) Does the game favor Player P_1?

12.4.5 Find the optimal strategies for Players P_1 and P_2 and the value of the game

$$\begin{array}{cc} & P_2 \\ P_1 & \begin{bmatrix} 3 & 7 & 1 \\ 2 & 6 & 8 \\ 1 & 3 & 4 \end{bmatrix} \end{array}$$

12.4.6 Determine the optimal strategies for Players P_1 and P_2 and the value of the following game, if possible:

$$\begin{array}{cc} & P_2 \\ P_1 & \begin{bmatrix} 3 & 4 & -4 & 6 & 0 \\ -3 & 5 & -2 & 7 & 10 \end{bmatrix} \end{array}$$

12.5 REDUCING MATRIX GAMES TO SYSTEMS OF EQUATIONS

The aim of the present section is to show that a matrix game can be reduced to a problem in linear programming. We can then solve matrix games by applying the techniques introduced in Chapter 10. We shall illustrate the procedure by considering a specific example. Suppose that we have a two-person zero-sum matrix game given by

$$\begin{array}{cc} & P_2 \\ P_1 & \begin{bmatrix} 4 & -2 \\ -6 & 8 \end{bmatrix} \end{array}$$

Player P_1 wants to utilize the best strategy to maximize his expected payoff E. Player P_2 wants to choose his strategies so that he will minimize the expected payoff. Thus P_1 and P_2 are the maximizer and minimizer of E, respectively. Let $[p_1 \ \ p_2]$ and $\begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$ be the strategy probabilities of Players P_1 and P_2, respectively. Proceeding as in the previous section, we can write

$$[p_1 \ \ p_2]\begin{bmatrix} 4 & -2 \\ -6 & 8 \end{bmatrix} \geq E$$

which can be stated as

$$4p_1 - 6P_2 \geq E$$
$$-2p_1 + 8P_2 \geq E$$

restricted by the fact that we must have

$$p_1 \geq 0. \quad p_2 \geq 0, \quad Ep_1 + p_2 = 1.$$

There is new restriction here that E must be positive for this procedure to work. However, if E is not positive, we can make it so by adding an appropriate constant to every term in the inequalities without changing the character of the problem.

Thus, P_1 must select P_1 and P_2 to maximize his expected payoff **E** subject to the following constraints:

$$4p_1 - 6P_2 \geq E$$
$$-2p_1 + 8P_2 \geq E$$
$$p_1 + p_2 = 1.$$
$$p_1 \geq 0, \quad p_2 \geq 0, \quad E > 0.$$

To simplify the problem further, we divide the preceding inequalities and equality by E; that is,

$$4\frac{p_1}{E} - 6\frac{p_2}{E} \geq 1$$
$$-2\frac{p_1}{E} + 8\frac{p_2}{E} \geq 1$$
$$\frac{p_1}{E} - 6\frac{p_2}{E} = \frac{1}{E}$$
$$\frac{p_1}{E} \geq 0, \quad \frac{p_2}{E} \geq 0.$$

Let $z_1 = p_1/E$ and $z_2 = p_2/E$. Now, the object of the preceding inequalities is to minimize

$$z_1 + z_2 = \frac{1'}{E}$$

subject to the constraints

$$4z_1 - 6z_2 \geq 1$$
$$-2z_1 + 8z_2 \geq 1$$
$$z_1 \geq 0, \quad z_2 \geq 0,$$

which is simply in the form of a linear program. Note that when we maximize E, it means that we minimize $1/E$; that is, increasing E implies that we are decreasing the value of $1/E$.

We begin by treating the constraints as if they were equalities. Solving them for z_1 and z_2 we obtain

$$z_1 = \frac{7}{10} \text{ and } z_2 = \frac{3}{10}.$$

You recall that

$$z_1 = \frac{p_1}{E} \text{ and } z_2 = \frac{p_2}{E}.$$

subject to

$$z_1 + z_2 = \frac{1}{E}.$$

Substituting the values of z_1 and z_2 in the above equation we have

$$\frac{7}{10} + \frac{3}{10} = \frac{1}{E}.$$

Solving this expression for the expected payoff, we obtain $E = 1.0$. Thus, substituting this value of E in

$$p_1 = z_1 E \text{ and } p_2 = z_2 E,$$

we have

$$p_1 = \frac{7}{10}(1) = \frac{7}{10} \text{ and } p_2 = \frac{3}{10}(1) = \frac{3}{10}$$

Thus, Player $\boldsymbol{P_1}$ must select the first row of the matrix game with probability 7/10 and the second row with probability 3/10 so as to maximize his gain.

Similarly, Player $\boldsymbol{P_2}$'s objective to select the strategy probabilities $\boldsymbol{q_1}$ and $\boldsymbol{q_2}$ so as to minimize the expected payoff. That is,

$$\begin{bmatrix} 4 & -2 \\ -6 & 8 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \le E,$$

which can be written as

$$\begin{aligned} \mathbf{4}q_1 - \mathbf{2}q_2 &\le E \\ -\mathbf{6}q_1 + \mathbf{8}q_2 &\le E \end{aligned}$$

along with the restrictions

$$q_1 \ge \mathbf{0}, \quad q_2 \ge \mathbf{0}, \quad q_1 + q_2 = \mathbf{1}, \quad E > \mathbf{0}.$$

$\boldsymbol{P_2}$'s problem, then, is to choose q_1 and q_2 that will minimize the expected payoff E subject to the constraints

$$\begin{aligned} \mathbf{4}q_1 - \mathbf{2}q_2 &\le E \\ -\mathbf{6}q_1 + \mathbf{8}q_2 &\le E \\ q_1 - + q_2 &= \mathbf{1} \\ q_1 \ge \mathbf{0} \quad q_2 &\ge \mathbf{0}. \end{aligned}$$

Using arguments similar to those previously mentioned, and letting $\boldsymbol{\gamma_1 = q_1/E}$ and $\boldsymbol{\gamma_2 = q_2/E}$, we have reduced the game into the following linear **equations**:

$$\begin{aligned} \mathbf{4}\gamma_1 - \mathbf{2}\gamma_2 &\le \mathbf{1} \\ -\mathbf{6}\gamma_1 + \mathbf{8}\gamma_2 &\le \mathbf{1} \\ \gamma_1 \ge \mathbf{0}, \quad \gamma_2 &\ge \mathbf{0}. \end{aligned}$$

subject to minimizing $\boldsymbol{\gamma_1 + \gamma_2 = 1/E}$. Treating the first two inequalities as if they were equalities and solving them, we obtain $\boldsymbol{y_1 = 1/2}$ and $\boldsymbol{y_2 = 1/2}$.

Since we know that $\boldsymbol{E = 1}$, we can obtain the strategy probabilities:

$$q_1 = y_1 E \quad \text{or} \quad q_1 = \frac{\mathbf{1}}{\mathbf{2}}(\mathbf{1}) = \frac{\mathbf{1}}{\mathbf{2}}$$

and

$$q_2 = y_2 E \quad \text{or} \quad q_2 = \frac{\mathbf{1}}{\mathbf{2}}(\mathbf{1}) = \frac{\mathbf{1}}{\mathbf{2}}.$$

Note that the restriction that $\boldsymbol{y_1 + y_2 = \frac{1}{E}}$ or $\boldsymbol{\frac{1}{2} + \frac{1}{2} = 1}$ is satisfied. Thus, Player $\boldsymbol{P_2}$ must select columns one and two of the matrix game with equal probability.

Furthermore, we check our results by obtaining the value of the game as follows:

$$\begin{aligned} E &= \begin{bmatrix} \frac{7}{10} & \frac{3}{10} \end{bmatrix} \begin{bmatrix} 4 & -2 \\ -6 & 8 \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \\ &= \mathbf{1}. \end{aligned}$$

Therefore, the optimal strategies for Players $\boldsymbol{P_1}$ and $\boldsymbol{P_2}$ are [**7/10 3/10**] and [**1/2 1/2**], respectively. The corresponding value of the game is 1 and, since it is positive, it favors $\boldsymbol{P_1}$.

PROBLEMS

12.5.1 Find the optimal strategies for Players P_1 and P_2, in the following games:

$$\text{(a)}\ \begin{array}{cc} & P_2 \\ P_1 & \begin{bmatrix} -2 & 8 \\ 10 & 6 \end{bmatrix} \end{array} \quad \text{(b)}\ \begin{array}{cc} & P_2 \\ P_1 & \begin{bmatrix} 4 & -5 \\ -3 & 2 \end{bmatrix} \end{array}$$

12.5.2. Reduce the matching pennies game given below into a system of linear equations and find the optimal stratiegies for Players P_1 and P_2. In the game of matching pennies, consider the following situation: If Player P_1 turns up T and P_2 calls T, P_1 pays P_2 \$5. However, if Player P_2 calls H, then P_2 must pay P_1 \$8. Now if P_1 turns up H and P_2 calls T, then P_2 must pay P_1 \$10, but if P_2 calls H, then P_1 must pay P_2 \$12.

12.5.3. Solve the following matrix game:

$$\begin{array}{cc} & P_2 \\ P_1 & \begin{bmatrix} 6 & 8 & -8 & 12 & 0 \\ -6 & 10 & -4 & 14 & 20 \end{bmatrix} \end{array}$$

12.5.4. Find the optimal strategies for Players P_1 and P_2, of the matrix game

$$\begin{array}{cc} & P_2 \\ P_1 & \begin{bmatrix} 4 & -3 & 2 \\ 5 & 4 & 3 \\ 8 & -2 & 8 \end{bmatrix} \end{array}$$

by reducing it into a system of linear equations.

Hint: Begin by employing the row and column dominance relation.

CRITICAL THINKING AND BASIC EXERCISE

12.1 Find the best strategies for Players P_1 and P_2, in the matrix games given by:

$$\text{(a)}\ \begin{array}{ccc} & & P_2 \\ & & \begin{array}{cc} \gamma_1 & \gamma_2 \end{array} \\ P_1 & \begin{array}{c} \beta_1 \\ \beta_2 \end{array} & \begin{bmatrix} -1 & 4 \\ -3 & 2 \end{bmatrix} \end{array} \quad \text{(b)}\ \begin{array}{ccc} & & P_2 \\ & & \begin{array}{ccc} \gamma_1 & \gamma_2 & \gamma_3 \end{array} \\ P_1 & \begin{array}{c} \beta_1 \\ \beta_2 \\ \beta_3 \end{array} & \begin{bmatrix} -10 & 5 & -8 \\ 4 & 2 & 1 \\ 0 & -4 & -1 \end{bmatrix} \end{array}$$

12.2 Find the point of equilibrium of the following matrix games:

$$\text{(a)}\ \begin{bmatrix} 4 & -4 & 4 \\ -3 & -3 & -3 \\ 5 & -5 & 5 \end{bmatrix} \quad \text{(b)}\ \begin{bmatrix} 2 & -4 & 1 & -3 \\ 0 & 1 & 6 & -5 \\ 7 & 1 & 4 & -1 \end{bmatrix}$$

12.3 Are the following matrix games strictly determined?

$$\text{(a)}\begin{bmatrix} 4 & 12 & 10 \\ 14 & 8 & 2 \\ -6 & 4 & -8 \end{bmatrix} \quad \text{(b)}\begin{bmatrix} 2 & 3 & 4 & 5 & 6 \\ -1 & -2 & 1 & 2 & 3 \end{bmatrix}$$

12.4 Construct a 3 × 3 matrix game, the point of equilibrium of which is 5.

12.5 Construct a zero-sum two-person matrix game in which the payoff to P_1 is negative; that is, a gain for Player P_2.

12.6 Find the expected payoff of the matrix game given by

$$\begin{array}{ccc} & & P_2 \\ & & \begin{array}{cc} \gamma_1 & \gamma_2 \end{array} \\ P_1 & \begin{array}{c} \beta_1 \\ \beta_2 \end{array} & \begin{bmatrix} -10 & 8 \\ 5 & -8 \end{bmatrix} \end{array}$$

Player P_1 chooses his strategies with equal probability and Player P_2 selects strategy y_1 three-fourths of the time and strategy y_2 one-fourth of the time.

12.7 Determine the expected payoff for the matrix game given by

$$\begin{array}{cc} & P_2 \\ P_1 & \begin{bmatrix} 7 & 3 & -2 \\ -4 & 5 & 4 \\ 6 & 8 & 10 \end{bmatrix} \end{array}$$

for the strategy probabilities [**1/2 1/4 1/4**] and $\begin{bmatrix} {}^1/_6 \\ {}^1/_3 \\ {}^1/_2 \end{bmatrix}$ for Players P_1 and P_2, respectively.

12.8 Determine the optimal strategies for Players P_1 and P_2, and the value of the following games:

(a) $\begin{array}{cc} & P_2 \\ P_1 & \begin{bmatrix} 4 & -3 \\ -2 & 10 \end{bmatrix} \end{array}$ (b) $\begin{array}{cc} & P_2 \\ P_1 & \begin{bmatrix} 1 & -2 \\ -1 & 2 \end{bmatrix} \end{array}$

12.9 What are the optimal strategies for Players P_1 and P_2, and the value of the game:

$$\begin{array}{cc} & P_2 \\ P_1 & \begin{bmatrix} -8 & 4 & 5 \\ 10 & -6 & -5 \\ -9 & -7 & -6 \end{bmatrix} \end{array} ?$$

12.10 Find the optimal strategies for Players P_1 and P_2, and the value of the following matrix game:

$$\begin{array}{cc} & P_2 \\ P_1 & \begin{bmatrix} 7 & -1 & 6 & 10 & 12 \\ 14 & 5 & -2 & 6 & 8 \end{bmatrix} \end{array}$$

12.11 Find the optimal strategies for Players P_1 and P_2, in the following matrix games by reducing the games into linear programming problems:

(a) $\begin{array}{cc} & P_2 \\ P_1 & \begin{bmatrix} -8 & 4 \\ 10 & -6 \end{bmatrix} \end{array}$ (b) $\begin{array}{cc} & P_2 \\ P_1 & \begin{bmatrix} -8 & 12 \\ 14 & -6 \end{bmatrix} \end{array}$

12.12 Reduce the following matrix game into a linear programming problem and then obtain the optimal strategies and value of the game:

$$\begin{array}{cc} & P_2 \\ P_1 & \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \end{array}$$

SUMMARY OF IMPORTANT CONCEPTS

Definitions:

12.1.1. **Game theory** is concerned with the analysis of human behavior in conflict situations. In our brief introduction to the subject, we have considered only games between two persons (players); however, they may involve any number of players, teams, companies and so forth.

12.1.2. The states of the game are given in the form of a matrix called the **game matrix** or **payoff matrix**.

12.2.1 A **two-person zero-sum game** is a game played by two opponents with opposing interest and such that the payoff to one player is equal to the loss of the other.

12.2.2. A **zero-sum** games such that the loss or gain by Player P_1 is equal to the gain or loss by Player P_2.

12.3.1 A matrix game is said to be **strictly determined** if and only if there is an entry in the payoff matrix that is the smallest entry in its row and also the largest entry in its column. This entry is called the **saddle point** or **equilibrium point** and is the value of the game.

12.4.1. The strategy probabilities $\{p_1,p_2\}$ and $\{q_1,q_2\}$, as defined above are called the **optimal strategies** for Players P_1 and P_2, respectively. The expected payoff E is called the value of the game.

Rules:

12.3.1 The search for the saddle point of a matrix game, if one indeed exists, can be made in three steps:

Step 1 Write the minimum entry of each row at the end of that row and write the maximum entry of each column at the end of the column.

Step 2 Identify the *maximin strategy* for Player P_1 by circling the largest of the row minimum. Similarly, circle the smallest of the column maximum to identify the *minimax strategy* for Player P_2.

Step 3 The game has a saddle point if and only if the maximin and minimax are equal to the same number, and that number also appears at the intersection in the matrix of maximin (row) and minimax (column).

REVIEW TEST

1. Define **game theory**.
2. What is another name for **game matrix**?
3. A game played by two opponents with opposing interest and such that the payoff to one player is equal to the loss of the other is referred to by what name?
4. What is meant by **strictly determined**?
5. What is the value of the two-person zero-sum game, the payoff matrix of which is given by $\begin{bmatrix} -2 & 3 & 4 \\ 6 & -1 & 5 \end{bmatrix}$
6. Find the best strategies for Players P_1 and P_2 which will lead to the value of the game if the payoff matrix is given by $\begin{bmatrix} -4 & 1 & 5 \\ 5 & -4 & 6 \\ 7 & 3 & 8 \end{bmatrix}$.
7. Construct a **3×3** matrix game with a value of $5.
8. Construct a **3×3** matrix game with a negative value.
9. Obtain the expected payoff of the matrix game: $\begin{bmatrix} -5 & 7 \\ 4 & -3 \end{bmatrix}$

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